Frobenius Divisibility, part 1

Theorem (Frobenius):

$\varphi: G \to GL(V)$ irreducible representation

$d := \dim V$, $n := |G|$

$\Longrightarrow$ $d \mid n$

Idea: $\varphi: G \to GL(V)$ irrep, $\dim V = d$.

Define: $T_g := \sum_{x \in Cl(g)} \varphi_x \in Hom_G(\varphi, \varphi)$

$g \in G$

$[\ \varphi_a T_g \varphi_a^{-1} = T_g \quad \forall a \in G\ ]$

φ irreducible, Schur $\Rightarrow$ $T_g = \lambda_g I$

and $\lambda_g = \dfrac{|Cl(g)|}{d} \chi(g)$

$[\quad Tr(T_g) = |Cl(g)|\, \chi(g) = \lambda_g d\ ]$

$$S := \sum_{g\in G}{}' \chi(g^{-1})\varphi_g = \sum_{x_i}{}' \chi(x_i^{-1}) T_{x_i}$$

$\{x_i\}$ = set of representatives of conjugacy classes of G

$$\mathrm{Tr}(S) = \frac{n}{n}\sum_g{}' \underbrace{\chi(g^{-1})}_{\overline{\chi(g)}}\chi(g) = n\cdot\langle\chi,\chi\rangle = n$$

$$= \sum_{x_i}{}' \chi(x_i^{-1})\,\mathrm{Tr}(\lambda_{x_i} I) = \sum_{x_i}{}' \chi(x_i^{-1})\lambda_{x_i} d$$

WTS an integer

$$\boxed{\frac{n}{d} = \sum_{x_i}\chi(x_i^{-1})\lambda_{x_i}}$$

Let R be the subgroup of $(\mathbb{C},+)$ generated by the set of elements

$$\{\underline{\zeta^i\lambda_g} \mid 0\le i<n,\ g\in G\}$$

where $\zeta = e^{2\pi i/n}$ $\quad n=|G|$

Claim: $\dfrac{n}{d} = \sum_{x_i}{}' \chi(x_i^{-1})\lambda_{x_i} \in R$

$$\chi(g) = \mathrm{Tr}(\varphi_g) = \sum_j \zeta^{m_j}$$

$$\varphi_g \sim \mathrm{diag}(\zeta^{m_1}, \ldots, \zeta^{m_d})$$

$$o(g) \mid |G| = n$$

<u>Claim</u>: $\frac{n}{d} \in R \cap \mathbb{Q} = \mathbb{Z}$ $\Leftarrow$

$$\Rightarrow \quad d \mid n$$